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A PRIORI ESTIMATES FOR SECOND ORDER ELLIPTIC EQUATIONS AND THEIR APPLICATIONS

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ABSTRACT

We study the equation $(\lambda + H)u = f$ where H is a self-adjoint operator associated with the Dirichlet form in $L^2(\mathbb{R}^d, \rho dx)$. A priori estimates of the first and the second order derivatives of solutions are obtained under minimal restrictions on the coefficients of the operator and measure. As a consequence we give a criterion of the essential self-adjointness of the operator $H \restriction C_0^\infty(\mathbb{R}^d)$ with non-smooth coefficients.

In this paper we are concerned with linear elliptic second order equations in weighted L^2-spaces. These equations are connected with operators associated with the Dirichlet forms which has been studied extensively in connection with applications in quantum field theory and theory of Markov processes [1], [2]. We study the smoothness of solutions of the equation

$$(\lambda + H)u = f \ , \quad \lambda > 0 \ .$$

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Here H is an operator in $L^2(\mathbb{R}^d, \rho(x)dx)$ associated with the closure of the form

$$h(u,v) = \int_{\mathbb{R}^d} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \rho(x)dx \equiv < a\nabla u, \nabla v > , \tag{1}$$
$$\mathcal{D}(h) = C_0^\infty(\mathbb{R}^d) .$$

The main result of our work is the estimates of the first and the second order derivatives of the bounded solutions under minimal restrictions on the coefficients and measure. In the special case $a_{ij} = \delta_{ij}$ (Kroneker symbol) the estimates of the same kind were obtained in [3], for the corresponding parabolic equation in [4] where the infinite dimensional case was considered. The above mentioned estimates enable us to obtain a criterion of the essential self-adjointness of the operator $H \restriction C_0^\infty(\mathbb{R}^d)$ which improves criteria known so far (see [5]).

The following notations are used below

$$L^p \equiv L^p(\mathbb{R}^d, \rho dx), \quad \|\cdot\|_p \text{ is norm in } L^p ,$$

$$< f,g > \equiv \int_{\mathbb{R}^d} f(x)g(x)\rho dx, \sum_i \equiv \sum_{i=1}^{d} , \ \nabla_i \equiv \frac{\partial}{\partial x_i} .$$

C_b is the set of all uniformly continuous bounded functions.
C_b^∞ is the set of all infinitely differentiable functions bounded with their derivatives. β is the vector of logarithmic derivative of the measure ρdx, so $\beta_i = \frac{\nabla_i \rho}{\rho}$ in the sense of distributions.

$A \restriction \mathcal{D}$ means the restriction of the operator A on the set $\mathcal{D}$, $A_{\mathcal{B}}^{\sim}$ means the closure of the operator A in the space $\mathcal{B}$.

We assume further $\rho(x) > 0$ for almost every $x \in \mathbb{R}^d$,

$$\rho \in L^1(\mathbb{R}^d), \quad \rho a_{ij} \in L^1_{loc}(\mathbb{R}^d), \ \forall\, i,j = 1,2,\ldots,d,$$

$$\beta_i \in L^2_{loc}(\mathbb{R}^d, \rho dx), \quad \forall \ i = 1,2,\ldots,d,$$

$$\mathbb{R}^d \ni x \longrightarrow a_{ij}(x) \in \mathbb{R}^1 , \quad \forall\, i,j = 1,2,\ldots,d, \tag{2}$$

$a_{ij}(x) = a_{ji}(x)$ $\forall\, i,j = 1,2,\ldots,d$ and for almost every $x \in \mathbb{R}^d$

$$\sum_i \xi_i^2 \le \sum_{i,j} a_{ij}(x)\xi_i\xi_j \le \Lambda \sum_i \xi_i^2, \quad \forall\, \xi \in \mathbb{R}^d .$$

All the functions are supposed to be real.

It is well-known that form (1) is closable under conditions (2) (see [6]).

We begin our consideration from the solutions of the equations with smooth coefficients

$$\lambda U_n - \sum_{i,j}(\nabla_i + \beta_i^n)a_{ij}^n \nabla_j U_n = f \ , \quad \beta_i^n, a_{ij}^n, f \in C_b^\infty, \quad \lambda > 0 \ . \tag{3}$$

Let

$$A_n = (-\sum_{i,j}(\nabla_i + \beta_i^n)a_{ij}^n \nabla_j \restriction C_0^\infty)^\sim_{C_b \to C_b} \ .$$

One can see from the maximum principle that the following inequality holds true

$$\|U_n\|_\infty \le \lambda^{-1}\|f\|_\infty \ .$$

THEOREM 1: *Let U_n be a solution of the equation (3). Suppose that $\beta \in L^4$. Then*

$$\begin{aligned} &\|\nabla U_n\|_4^4 \le C(\|f\|_\infty^4 + \|f\|_2^4), \\ &\sum_{i,j}\|\nabla_i\nabla_j U_n\|_2^2 \le C(\|f\|_\infty^2 + \|f\|_2^2), \end{aligned} \tag{4}$$

where

$$\begin{aligned} C = & C(\lambda, d)(1 + \|\beta^n\|_4^4 + \|\beta\|_4^4 + \|\sum_j(\sum_i |a_{ij}^n\beta_i|)^2\|_2^2 \\ & + \|\sum_j(\sum_i |a_{ij}^n\beta_i^n|)^2\|_2^2 + \max_{i,j,k}\|\nabla_k a_{ij}\|_4^4) \ . \end{aligned}$$

Proof: We follow the procedure in [7]. By the regularity theory $U_n \in C_b^\infty$. Multiplying both parts of the equality

$$\lambda U_n - \sum_{i,j}(\nabla_i + \beta_i^n)a_{ij}^n W_j = f \ , \quad W_j = \nabla_j U_n$$

by $\nabla_k^+ W_k$, $\nabla_k^+ = -\nabla_k - \beta_k$, yields after integration by parts

$$\begin{aligned} \lambda\|W\|_2^2 &+ \sum_{i,j,k} < \nabla_j W_k, a_{ij}^n \nabla_i W_k > + \sum_{i,j,k} < (\nabla_k a_{ij}^n)W_j, \ \nabla_i W_k > \\ &+ \sum_{i,j,k} < a_{ij}^n(\nabla_j W_k), \beta_i W_k > + \sum_{i,j,k} < (\nabla_k a_{ij}^n)W_j, \ \beta_i W_k > \\ &+ \sum_{i,j,k} < a_{ij}^n\beta_i^n W_j, \ \nabla_k W_k > + \sum_{i,j,k} < a_{ij}^n\beta_i^n W_j, \ \beta_k W_k > \\ = &\sum_k < f, \ \nabla_k^+ W_k > \ . \end{aligned} \tag{5}$$

We use the following notations:

$$I = \sum_{i,j,k} < \nabla_j W_k, a_{ij}^n \nabla_i W_k >, \quad \mathcal{T} = \sum_{i,j} < W_i a_{ij}^n W_j, \ |W|^2 > \ .$$

Let us estimate terms in (5):

$$\begin{aligned}
&\sum_{i,j,k} | < (\nabla_k a_{ij}^n) W_j, \ \nabla_i W_k > | \\
&\leq < \sum_j |W_j| (\sum_{i,k} (\nabla_k a_{ij}^n)^2)^{1/2}, (\sum_{i,k} (\nabla_i W_k)^2)^{1/2} > \\
&\leq < |W| (\sum_{i,j,k} (\nabla_k a_{ij}^n)^2)^{1/2} (\sum_{i,k} (\nabla_i W_k)^2)^{1/2} > \\
&\leq \alpha \mathcal{T} + \gamma I + C_{\alpha,\gamma} \| \sum_{i,j,k} (\nabla_k a_{ij}^n)^2 \|_2^2, \quad a > 0, \quad \gamma > 0 \, , \\
&\sum_{i,j,k} | < a_{ij}^n \nabla_j W_k, \ \beta_i W_k > | \\
&\leq < (\sum_j (\sum_i |a_{ij}^n \beta_i|)^2)^{1/2}, (\sum_{j,k} (\nabla_j W_k)^2)^{1/2} |W| > \\
&\leq \alpha \mathcal{T} + \gamma I + C_{\alpha,\gamma} \| \sum_j (\sum_i |a_{ij}^n \beta_i|)^2 \|_2^2 \ , \\
&\sum_{i,j,k} | < (\nabla_k a_{ij}^n) W_j, \ \beta_i W_k > | \\
&\leq < |\beta| (\sum_{i,j,k} (\nabla_k a_{ij}^n)^2)^{1/2}, (\sum_{j,k} (W_j W_k)^2)^{1/2} > \\
&\leq \alpha \mathcal{T} + C_\alpha \|\beta\|_4^4 + \frac{1}{4} \ \| \sum_{i,j,k} (\nabla_k a_{ij}^n)^2 \|_2^2 \ , \\
&\sum_{i,j,k} | < a_{ij}^n \beta_i^n W_j, \ \nabla_k W_k > | \leq \alpha \mathcal{T} + \gamma I + C_{\alpha,\gamma,d} \| \sum_j (\sum_i |a_{ij}^n \beta_i^n|))^2 \|_2^2 \ , \\
&\sum_{i,j,k} | < a_{ij}^n \beta_i^n W_j, \ \beta_k W_k > | \leq \alpha \mathcal{T} + C_\alpha \|\beta\|_4^4 + \frac{1}{4} \ \| \sum_j (\sum_i |a_{ij}^n \beta_i^n|)^2 \|_2^2 \ , \\
&\sum_k | < f, \ \nabla_k W_k > | \leq \gamma I + C_{d,\gamma} \|f\|_2^2 \ , \\
&\sum_k | < f, \ \beta_k W_k > | \leq \alpha \mathcal{T} + C_\alpha \|\beta\|_4^4 + \frac{1}{2} \ \|f\|_2^2 \ .
\end{aligned}$$

Finally we obtain the following inequality:

$$
\begin{aligned}
(6)\qquad \lambda\|W\|_2^2 + I \le& \delta\mathcal{T} + C_\delta(\|\sum_{i,j,k}(\nabla_k a_{ij}^n)^2\|_2^2 + \|\beta\|_4^4 + \|\sum_j(\sum_i |a_{ij}^n\beta_i|)^2\|_2^2) \\
&+ C_{\delta,d}(\|\sum_j(\sum_i |a_{ij}^n\beta_i^n|)^2\|_2^2 + \|f\|_2^2).
\end{aligned}
$$

Now let us prove the inverse estimate using the equation (3):

$$
\begin{aligned}
\mathcal{T} =& \sum_{i,j} < W_i a_{ij}^n W_j |W|^2 >= -\sum_{i,j} < U_n, (\nabla_i + \beta_i) a_{ij}^n W_j |W|^2 > \\
=& < U_n, (\lambda U_n - f)|W|^2 > - \sum_{i,j} < U_n, |W|^2(\beta_i - \beta_i^n) a_{ij}^n W_j > \\
&- 2\sum_{i,j,k} < U_n, a_{ij}^n W_j W_k \nabla_i W_k > \\
\le& \|U_n(\lambda U_n - f)\|_\infty \|W\|_2^2 \\
(7)\qquad &+ \|U_n\|_\infty (\sum_{i,j} < W_i a_{ij}^n W_j, |W|^2 >)^{1/2} (\sum_{i,j} < |\beta_i - \beta_i^n|,\ |W|^2 >)^{1/2} \\
&+ 2\|U_n\|_\infty (\sum_{i,j,k} < \nabla_i W_k, a_{ij}^n \nabla_j W_k >)^{1/2} (\sum_{i,j} < W_i a_{ij}^n W_j, |W|^2 >)^{1/2} \\
\le& \frac{2}{\lambda}\ \|f\|_\infty^2 \|W\|_2^2 + \frac{1}{4}\ \mathcal{T} + \frac{1}{\lambda^2}\ \|f\|_\infty^2 \sum_{i,j} < (\beta_i - \beta_i^n) a_{ij}^n (\beta_j - \beta_j^n), |W|^2 > \\
&+ \frac{4}{\lambda^2}\ \|f\|_\infty^2 I + \frac{1}{4}\mathcal{T} \le \frac{1}{2}\mathcal{T} + \frac{4}{\lambda^2}\ \|f\|_\infty^2 I + \frac{2}{\lambda}\ \|f\|_\infty^2 \|W\|_2^2 \\
&+ \frac{1}{\lambda^2}\ \|f\|_\infty^2 I + \frac{1}{4\lambda^2}\|\sum_{i,j}(\beta_i - \beta_i^n) a_{ij}^n (\beta_j - \beta_j^n)\|_2^2 \cdot \|f\|_\infty^2.
\end{aligned}
$$

Estimates (6), (7) with $\delta = \lambda^2/20\|f\|_\infty^2$ imply the statement of the theorem.

Remark 1: Estimates (4) do not depend on the largest eigenvalue of the matrix (a_{ij}^n).

THEOREM 2: *Let* $a_{ij} \in L^\infty$, $\nabla_k a_{ij} \in L^4$, $\beta_i \in L^4$ $\forall\ i,j,k = 1,2,\ldots,d$. *Then* $(H \restriction C_0^\infty(\mathbb{R}^d))^\sim = (H \restriction C_0^\infty(\mathbb{R}^d))^*$ *(i.e. operator* $H \restriction C_0^\infty(\mathbb{R}^d)$ *is essentially self-adjoint).*

Proof: Let $f \in C_b^\infty$. Then $U_n \in C_b^\infty$ where U_n is a solution of the equation $(1 + A_n)U_n = f$ in the space C_b. We have the following equality in L^2:

$$
\begin{aligned}
(1 + H)U_n =& f + \sum_{i,j}(a_{ij}^n - a_{ij})\nabla_i\nabla_j U_n \\
&+ \sum_{i,j}(\nabla_i a_{ij}^n - \nabla_i a_{ij})\nabla_j U_n + \sum_{i,j}(\beta_i^n a_{ij}^n - \beta_i a_{ij})\nabla_j U_n\ .
\end{aligned}
$$

Choose sequences $\{a^n_{ij}\}$, $\{\beta^n_i\}$ such that

$$\sup_n \|a^n_{ij} - a_{ij}\|_\infty < \infty, \ \lim_n \|a^n_{ij} - a_{ij}\|_2 = 0, \ \lim_n \|\nabla_i a^n_{ij} - \nabla_i a_{ij}\|_4 = 0 \ ,$$

$$\lim_n \sum_i \|\beta^n_i a^n_{ij} - \beta_i a_{ij}\|_4 = 0 \ .$$

Using Theorem 1 we have

$$\lim_n \|(1+H)U_n - f\|_2 = 0.$$

So $Ran((1+H) \restriction C^\infty_b)^\sim = L^2$.

Let $w \in C^\infty_0$, $0 \le w \le 1$, $w(x) = 1$ if $|x| < 1$ and $w(x) = 0$ if $|x| > 2$, $w_n(x) = w(\frac{x}{n})$. It is easy to check directly that

$$\lim_n \|(1+H)w_n f - (1+H)f\|_2 = 0$$

for $\forall f \in C^\infty_b$.

Remark 2: In the special case $a_{ij} = \delta_{ij}$ this result was obtained in [4], where it was shown that condition $\beta \in L^4$ could not be improved.

THEOREM 3: *Let $A_n = (-\sum_i(\nabla_i + \beta^n_i)k^2\nabla_i \restriction C^\infty_b)^\sim_{C_b}$. Suppose that $\beta \in L^4$, $\beta^n \in C^\infty_b, k \in C^\infty_b$. Let U_n be a solution of the equation $(\lambda + A_n)U_n = f$, $\lambda > 0$, $f \in C^\infty_b$. Then*

$$\|k\nabla U_n\|_4^4 \le C\|f\|^4_\infty,$$

$$\sum_{i,j} \|k\nabla_i\nabla_j U_n\|^2_2 \le C\|f\|^2_\infty,$$

where $C = C(\lambda)(1 + \|k\beta\|^4_4 + \|k\beta^n\|^4_4 + \|\nabla k\|^4_4)$, $C(\lambda) > 0$.

Proof: Multiply both parts of the equation $(\lambda + A_n)U_n = f$ by $\nabla^+_j k^2 W_j$, $W_j = \nabla_j U_n$ in L^2. Using the equation we have

$$\begin{aligned}
&\lambda\|kW\|^2_2 + I + 4\sum_{i,j} < k^2\nabla_i W_j, (\nabla_i k)kW_j > + \sum_{i,j} < k^2\nabla_i W_j, \beta_i k^2 W_j > \\
(8) \quad &+ 4\|(\nabla k)kW\|^2_2 + 2 < (\nabla k)kW, \ k\beta > + < \beta^n k^2 W, (f - \lambda U_n) > \\
&+ < \beta^n k^2 W, (\beta^n - \beta)k^2 W > = < f, (f - \lambda U_n) > + < f, (\beta^n - \beta)k^2 w >
\end{aligned}$$

where $I = \sum_{i,j} \|k^2\nabla_i W_j\|^2_2$.

Using Hölder's and Cauchy's inequalities to estimate terms in (8) we have

$$\lambda\|kW\|_2^2 + I \le \delta\|kW\|_4^4 + C_\lambda(\|\nabla k\|_4^4 + \|k\beta\|_4^4 + \|k\beta^n\|_4^4) \times (\|f\|_1 + \|f\|_\infty)\|f\|_\infty^3\,, \quad \delta > 0,\ C_\lambda > 0\,. \tag{9}$$

Again using the equation after integration by parts we obtain

$$\|kW\|_4^4 = < U_n, k^2|W|^2(\lambda U_n - f) > + < U_n, k^2|W|^2(\beta^n - \beta)k^2 W > - 2 < U_n, (\nabla k)kW \cdot k^2|W|^2 > -2\sum_{i,j} < U_n, k^2 W_i W_j k^2 \nabla_i W_j > \,.$$

From the last equality

$$\|kW\|_4^4 \le \frac{4}{\lambda^2}\,\|f\|_\infty^2 I + C_\lambda(\|\nabla k\|_4^4 + \|k|\beta - \beta^n|\|_4^4)\|f\|_\infty^4\,.$$

Combining this inequality with (9) and choosing $\delta = \lambda^2/8\|f\|_\infty^2$ we obtain the statement of the theorem.

Remark 3: It should be pointed out that estimates in Theorem 3 do not depend on dimension.

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